2. Place Value

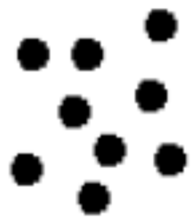
"The idea of expressing all quantities by nine figures whereby is imparted to them both an absolute value and one by position is so simple that this very simplicity is the very reason for our not being sufficiently aware how much admiration it deserves."
-Laplace

Book 2 - Place Value

SECTION 1

DOTS AND BOXES

Here are some dots; in fact there are nine of them:



Here are some boxes:

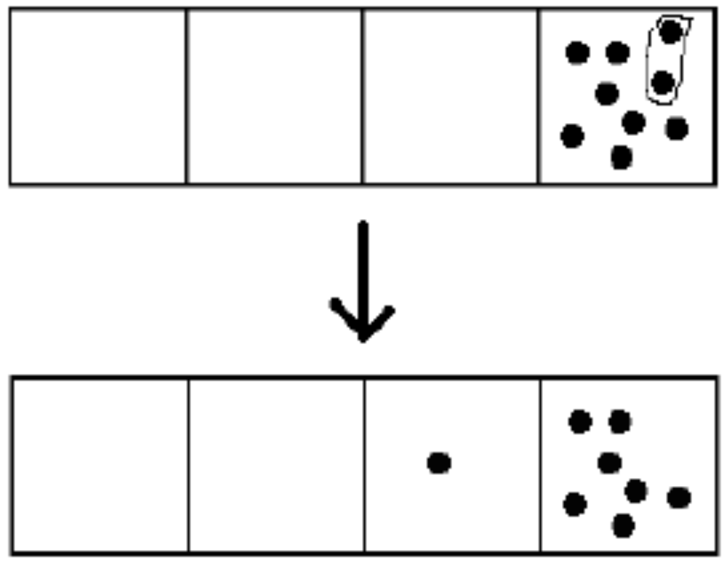
We're going to a play a game in which boxes explode dots and move them around. Here's our first rule:

The $1 \leftarrow 2$ Rule

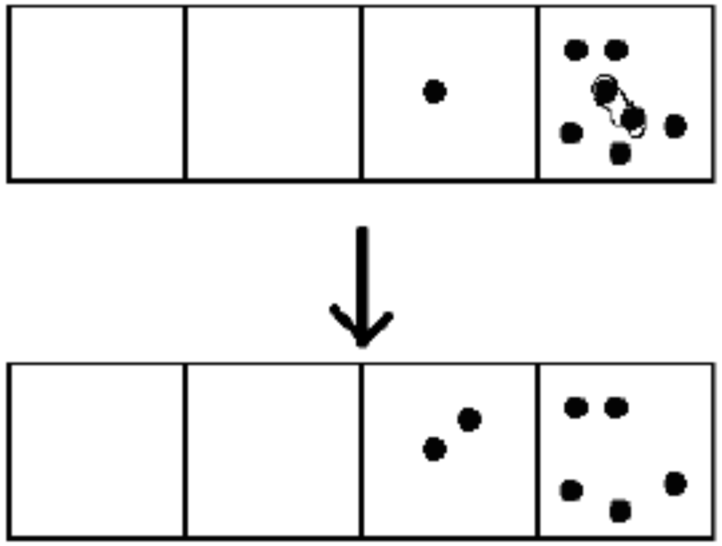
Whenever there are two dots in single box, they “explode," disappear, and become one dot in the box to the left.

Example 1.1 (Nine dots in the $1 \leftarrow 2$ system). We start by placing nine dots in the rightmost box.

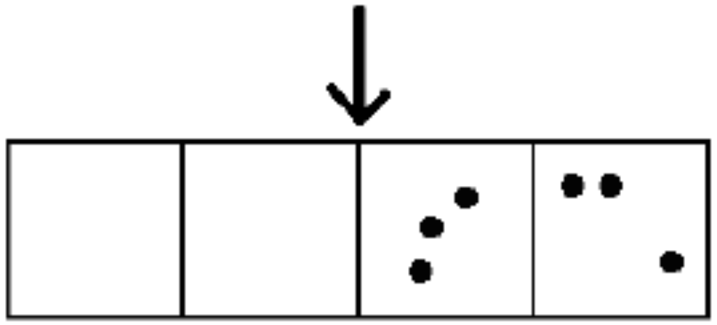
Two dots in that box explode and become one dot in the box to the left.



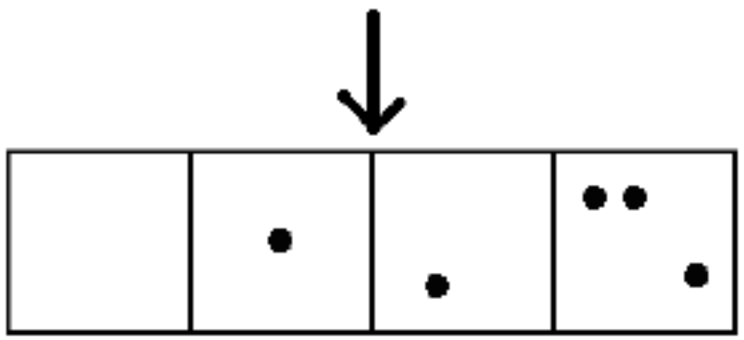
Two dots in that box explode and become one dot in the box to the left.



And again!

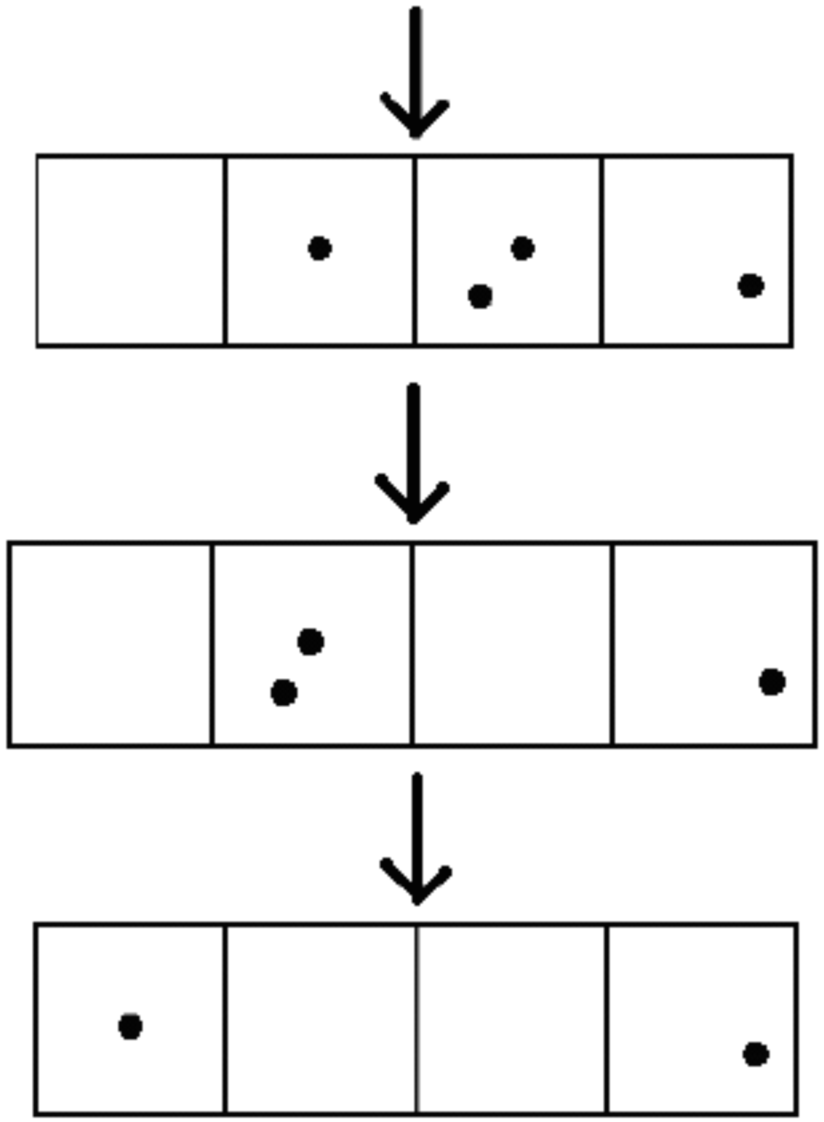


Hey, now we have more than two dots in the second box, so those can explode and move!



And the rightmost box still has more than two dots.

Keep going, until no box has two dots.

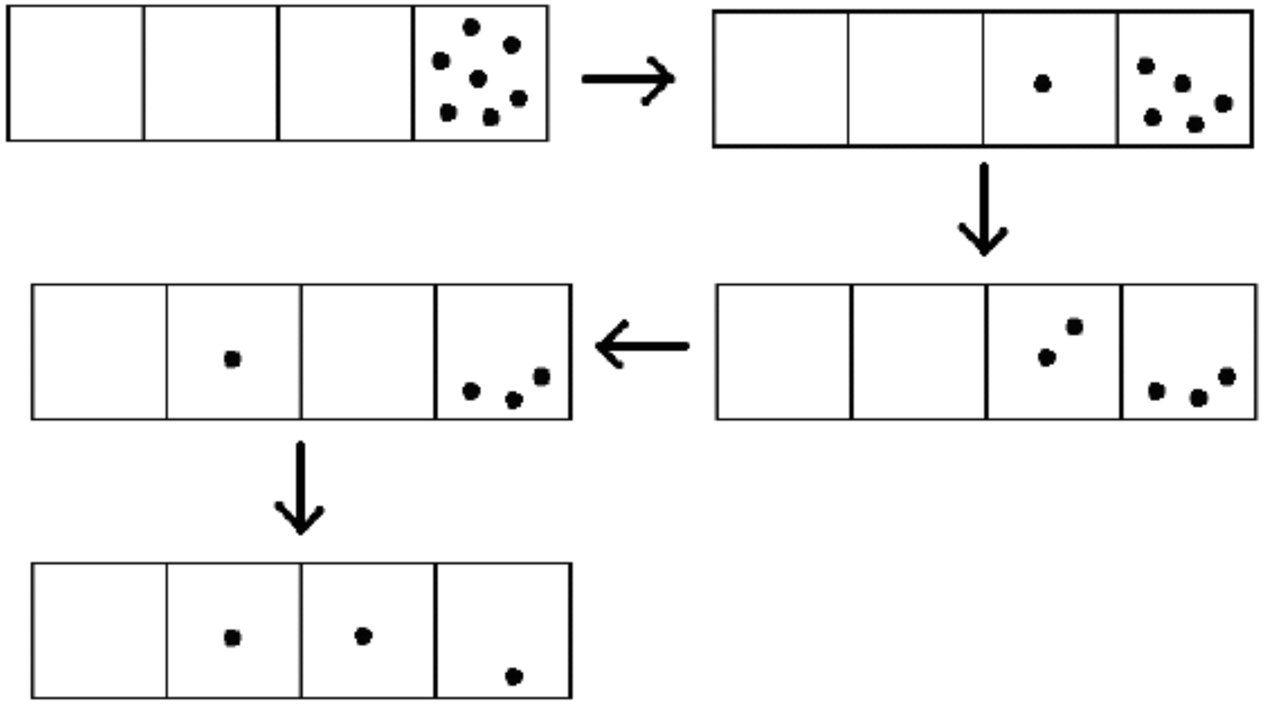


After all this, reading from left to right we are left with one dot, followed by zero dots, zero dots, and one final dot.

Solution (Example 1.1).

The $1 \leftarrow 2$ code for nine dots is: 1001

On Your Own.
Here's a diagram showing what happens for seven dots in a $1 \leftarrow 2$ box. Trace through the diagram, and circle the pairs of dots that “exploded" at each step.



Solution.
The $1 \leftarrow 2$ code for seven dots is: 111

Problem 1.
Note: In solving this problem, you don't need to draw on paper; that can get tedious! Maybe you could use buttons or pennies for dots and do this by hand. What could you use for the boxes?

a) Draw 10 dots in the right-most box and perform the explosions. What is the $1 \leftarrow 2$ code for ten dots?
b) Find the $1 \leftarrow 2$ code for thirteen dots.
c) Find the $1 \leftarrow 2$ code for six dots.
d) What number of dots has $1 \leftarrow 2$ code 101 ?

Think/Pair/Share.
After you worked on the problem, compare your answer with a partner. Did you both get the same code? Did you have the same process?

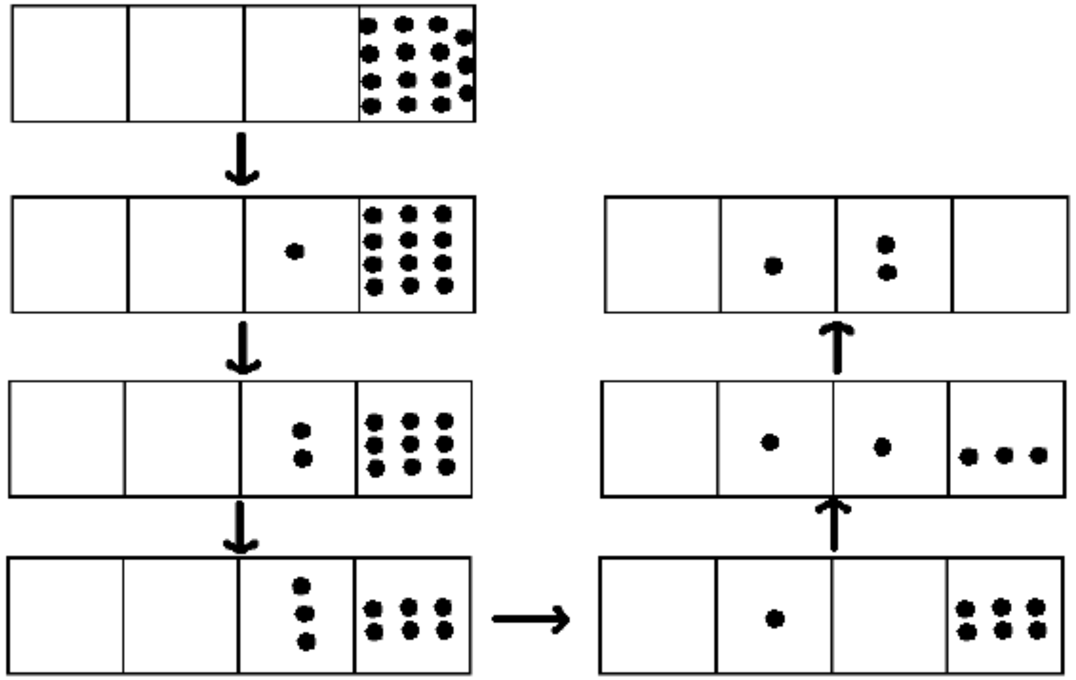
SECTION 2

OTHER RULES

Let's play the dots and boxes game, but change the rule.

The 1 ← 3 Rule
Whenever there are three dots in single box, they "explode," disappear, and become one dot in the box to the left.

Example 2.1 (Fifteen dots in the 1 ← 3 system). Here's what happens with fifteen dots:



Solution (Example 2.1).
The 1 ← 3 code for fifteen dots is: 120

Problem 2.
a) Show that the 1 ← 3 code for twenty dots is 202.
b) Show that the 1 ← 3 code for four dots is 11.
c) What is the 1 ← 3 code for thirteen dots?
d) What is the 1 ← 3 code for twenty-five dots?
e) What number of dots has 1 ← 3 code 1022?
f) Is it possible for a collection of dots to have 1 ← 3 code 2031? Explain your answer.

Problem 3.
a) Describe how the 1 ← 4 rule would work.
b) What is the 1 ← 4 code for the number thirteen?

Problem 4.
a) What is the 1 ← 5 code for the number thirteen?
b) What is the 1 ← 5 code for the number five?

Problem 5.
a) What is the 1 ← 9 code for the number thirteen?
b) What is the 1 ← 9 code for the number thirty?

Problem 6.

a) What is the $1 \leftarrow 10$ code for the number thirteen?

b) What is the $1 \leftarrow 10$ code for the number thirty-seven?

c) What is the $1 \leftarrow 10$ code for the number two hundred thirty-eight?

d) What is the $1 \leftarrow 10$ code for the number five thousand eight hundred and thirty-three?

Think/Pair/Share.

After you have worked on the problems on your own, compare your ideas with a partner. Can you describe what's going on in Problem 6 and why?

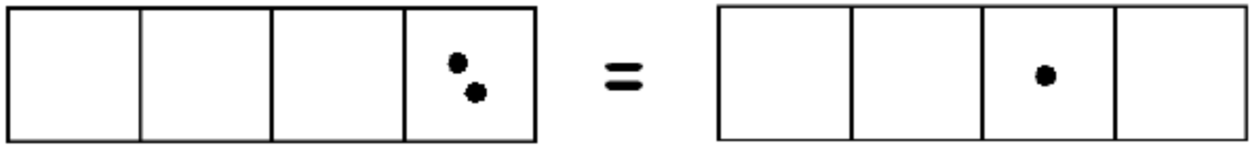
Section 3

Binary Numbers

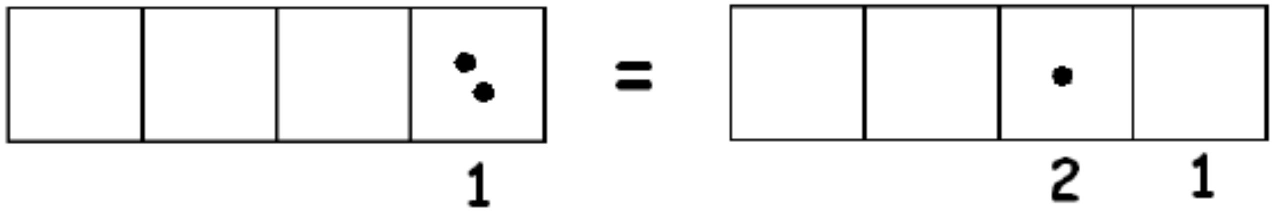
Let's go back to the 1 ← 2 rule for a moment:

The 1 ← 2 **Rule**
Whenever there are two dots in single box, they “explode," disappear, and become one dot in the box to the left.

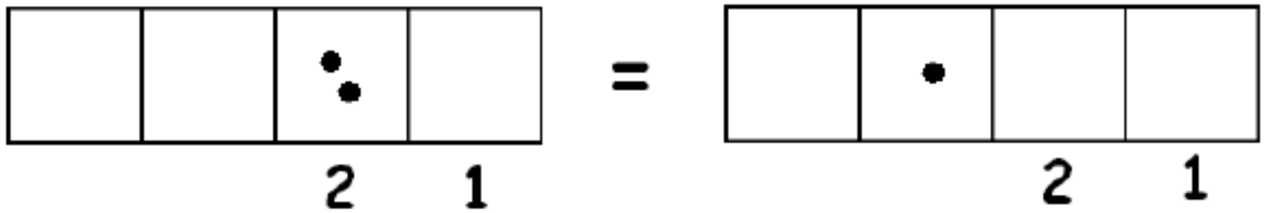
Two dots in the right-most box is worth one dot in the next box to the left.



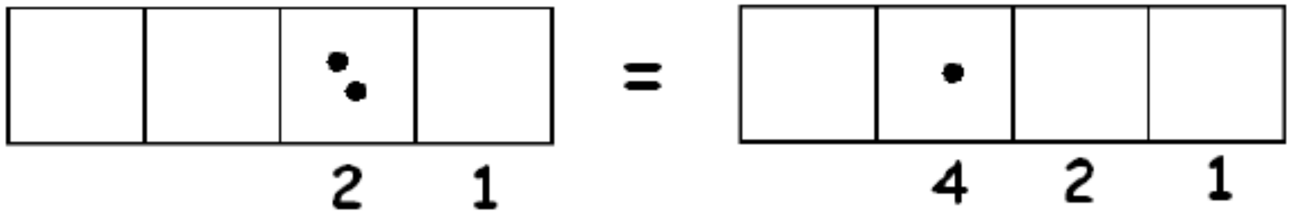
If each of the original dots is worth “one," then the single dot on the left must be worth two.



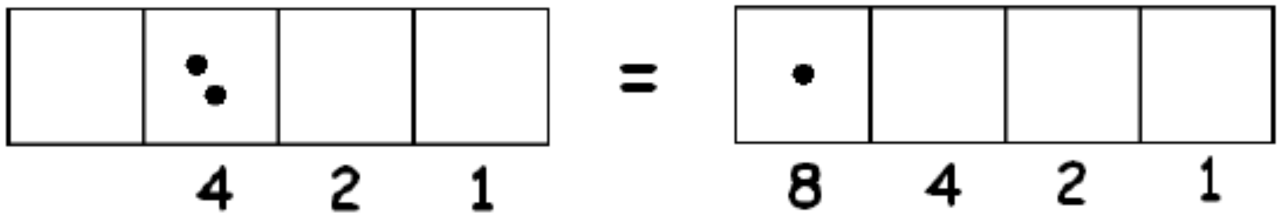
But we also have two dots in the box of value 2 is worth 1 dot in the box just to the left…



So that next box must be worth two 2s, which is four!



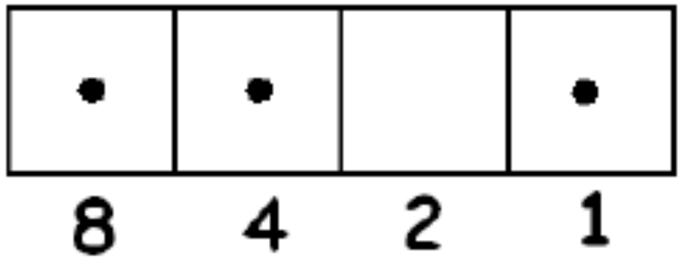
And two of these fours make eight.



Example 3.1 (Nine dots in the $1 \leftarrow 2$ system revisited). We said earlier that the $1 \leftarrow 2$ code for nine dots was 1001. Let's check:

$$8 + 1 = 9, \text{ so this works!}$$

We also said that thirteen has $1 \leftarrow 2$ code 1101. This is correct.



Yep! $8 + 4 + 1 = 13$

Problem 7.

a) If there were a box to the left of the 8 box, what would the value of that box be?

b) What would be the value of a box *two* spots to the left of the 8 box? Three spots to the left?

c) What number has $1 \leftarrow 2$ code 100101?

d) What is the $1 \leftarrow 2$ code for the number two hundred?

Definition 3.2. Numbers written in the $1 \leftarrow 2$ code are called *binary numbers* or *base two numbers*. (The prefix "bi" means "two.") From now on, when we want to indicate that a number is written in base two, we will write a subscript "two" on the number. So 1001_{two} means "the number of dots that has $1 \leftarrow 2$ code 1001," which we already saw was nine.

Important! When we read 1001_{two} we say "one zero zero one base two." We don't say "one thousand and one," because "thousand" is not a binary number.

Think/Pair/Share. Compare you work on problem 7 with a partner.

1) Your first goal: come up with a *general method* to find the number of dots represented by any binary number. Clearly describe your method. Test your method out on these numbers, and check your work by actually "unexploding" the dots.

$$1_{\text{two}} \qquad 101_{\text{two}} \qquad 1011_{\text{two}} \qquad 1111_{\text{two}} \qquad 1101101_{\text{two}}$$

2) Explain why binary numbers only contain the digits 0 and 1.

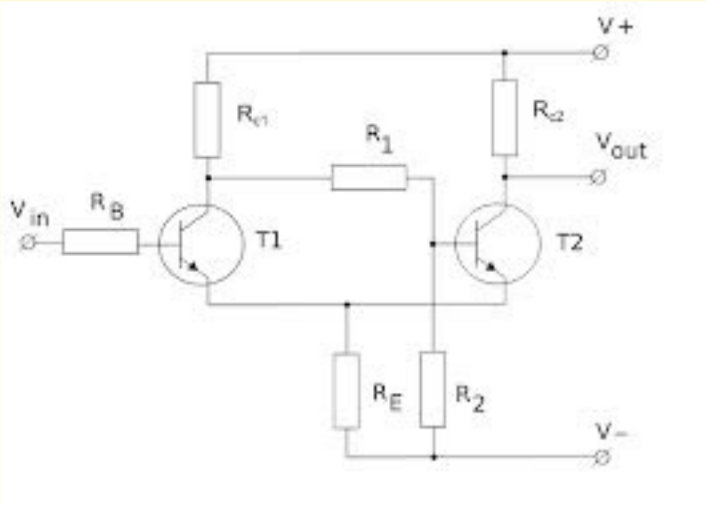
3) Here is a new (harder) goal: come up with a *general method* to find the binary number related to any number of dots *without actually going through the "exploding dot" process*. Clearly describe your method. Test your method out on these numbers, and find a way to check your work.

two dots $= ???_{\text{two}}$ seventeen dots $= ???_{\text{two}}$ sixty-four dots $= ???_{\text{two}}$

sixty-three dots $= ???_{\text{two}}$ one thousand dots $= ???_{\text{two}}$

Binary

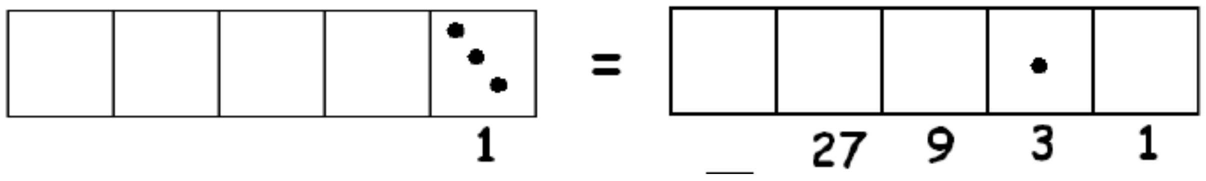
You probably realize by now that a number is an abstract concept with many representations. The standard decimal representation of a number is only one of these. For computers, numbers are always represented in binary. The basic units are transistors which are either on (1) or off (0). A transistor is said to store ***one bit*** of information. Eight bits make a byte and a typical home computer's central processing unit performs computations on registries that are each 8 bytes (64-bits). Using the $1 \leftarrow 2$ rule we can represent the numbers 0 through 18,446,744,073,709,551,615 with 64 bits.



SECTION 4

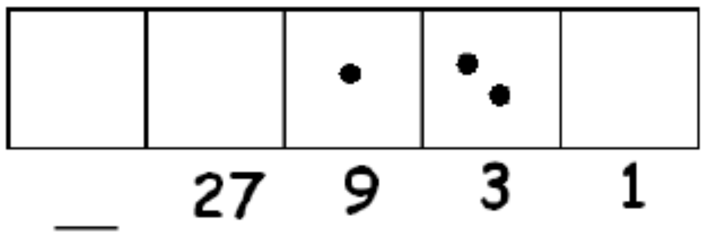
OTHER BASES

In the $1 \leftarrow 3$ system, three dots in one box is worth one dot in the box one spot to the left. This gives a new picture:



Each dot in the second box from the left is worth three ones. Each dot in the third box is worth three 3s, which is nine, and so on.

Example 4.1. We said that the $1 \leftarrow 3$ *code for fifteen is* 120. *We see that this is correct because* $1 \cdot 9 + 2 \cdot 3 + 0 \cdot 1 = 15$



Problem 8. Answer these questions about the $1 \leftarrow 3$ system.

a) What label should go on the box to the left of the 27 box?

b) What would be the value of a box two spots to the left of the 27 box?

c) What number has $1 \leftarrow 3$ code 21002?

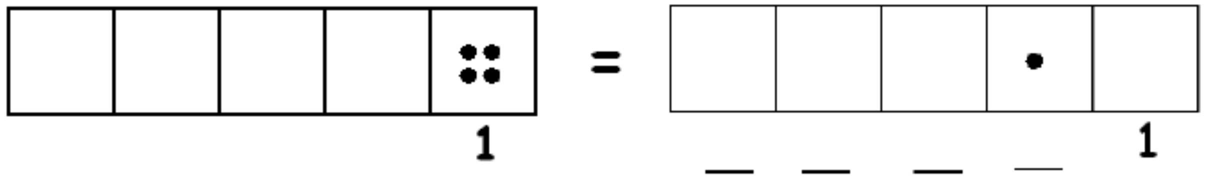
d) What is the $1 \leftarrow 3$ code for the number two hundred?

Problem 9. In the $1 \leftarrow 4$ system, four dots in one box are worth one dot in the box one place to the left.

a) What is the value of each box?

b) What is the $1 \leftarrow 4$ code for twenty-nine?

c) What number has $1 \leftarrow 4$ code 132?



Problem 10. In the $1 \leftarrow 10$ system, ten dots in one box are worth one dot in the box one place to the left.

a) What is the value of each box?

b) What is the $1 \leftarrow 10$ code for eight thousand four hundred and twenty-two?

c) What number has $1 \leftarrow 10$ code 95753?

d) When we write the number 7842 the "7" is represents what quantity? The "4" is four groups of what value? The "8" is eight groups of what value? The "2" is two groups of what value?

e) Why do human beings like the $1 \leftarrow 10$ system for writing numbers?

Definition 4.2. Numbers written in the $1 \leftarrow 3$ system are called *base three numbers*. Numbers written in the $1 \leftarrow 4$ system are called *base four numbers*. Numbers written in the $1 \leftarrow 10$ system are called *base ten numbers*. In general, numbers written in the $1 \leftarrow b$ system are called *base b numbers*.

In a base b number system, each place represents a power of b, which means b^k for some positive number k. Remember this means b multiplied by itself k times:

$$b^k = \underbrace{b \cdot b \cdots b}_{k \text{ times}}.$$

- The right-most place is the units or ones place. (Why is this a power of b?)
- The second spot is the "b" place. (In base 10, it's the tens place.)
- The third spot is the "b^2" place. (In base 10, that's the hundreds place, and $100 = 10^2$.)
- The fourth spot is the "b^3" place. (In base 10, that's the thousands place, and $1000 = 10^3$.)
- And so on… the n^{th} spot is the b^{n-1} place.

Notation: Whenever we're dealing with numbers written in different bases, we use a subscript to indicate the base so that there can be no confusion. So 102_{three} is a base three number, 222_{four} is a base four number, and 54321_{ten} is a base ten number. If the base is not written, we assume the number is expressed in base ten.

Think/Pair/Share.
1) Find the number of dots represented by:

$$102_{\text{three}}, \quad 222_{\text{four}}, \quad 54321_{\text{ten}}.$$

2) Represent nine dots in each base:

three, four, five, six, seven, eight, nine, and ten.

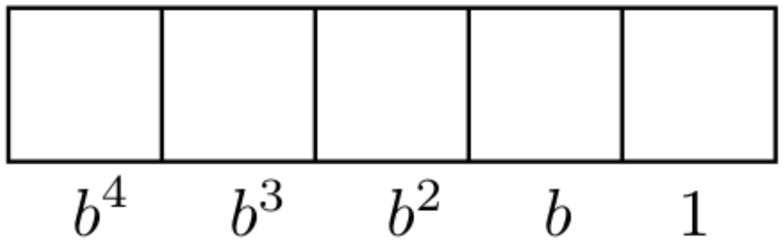
3) Which digits are used in the base two system? The base three system? The base four system? The base five system? The base six system? The base ten system?
4) What does the *base* tell you about the number system? (Think of as many answers as you can!)

Subsection 4.1

Base b to Base Ten

In Section 3, you were asked to come up with *general methods* to translate numbers from base two (binary) to base ten (our standard system). We're now going to describe some general methods for converting from base b to base ten, where b can represent any whole number bigger than one.

If the base is b, that means we're in a $1 \leftarrow b$ system. A dot in the right-most box is worth 1. A dot in the second box is worth b. A dot in the third box is worth $b \times b = b^2$, and so on.



So, for example, the number 10123_b represents

$$1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 2 \cdot b + 3 \cdot 1 \text{ dots},$$

because we imagine three dots in the right-most box (each worth one), two dots in the second box (each representing b dots), one dot in the third box (representing b^2 dots), and so on. That means we can just do a short calculation to find the total number of dots, without going through all the trouble of drawing the picture and "unexploding" the dots.

Example 4.3. Consider the number 123_{five}. *This represents*

$$1 \cdot 5^2 + 2 \cdot 5 + 3 = 25 + 10 + 3 = 38 \text{ dots}.$$

On the other hand, the number 123_{seven} represents

$$1 \cdot 7^2 + 2 \cdot 7 + 3 = 49 + 14 + 3 = 66 \text{ dots}.$$

Think/Pair/Share.

- Convert each number to base ten. Compare your answers with a partner to be sure you agree.

  $$18_{\text{nine}}, \quad 547_{\text{eight}}, \quad 3033_{\text{five}}, \quad 11011_{\text{three}}.$$

- Which number represents a greater amount of total dots:

  $$23{,}455{,}443_{\text{six}} \quad \text{or} \quad 23{,}455{,}443_{\text{eight}}?$$

  Justify your answer.

Subsection 4.2

Base Ten to Base b

In Section 3, you were also asked to come up with *general methods* to translate numbers from base ten to base two. We're now going to describe some general methods for converting from base ten to base b, where b can represent any whole number bigger than one.

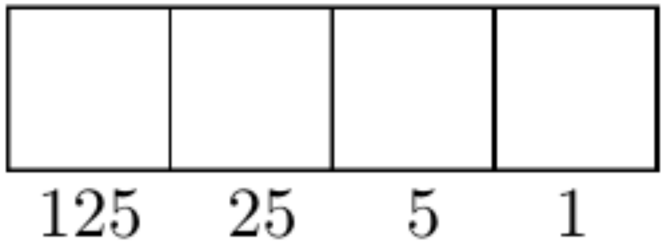
We'll work out an example, and then describe the general method.

Example 4.4. To convert 321 *to a base five number (without actually going through the tedious process of exploding* 321 *dots in groups of five):*

Find the largest power of five that is smaller than 321. We'll just list powers of five:

$$5^1 = 5, \quad 5^2 = 25, \quad 5^3 = 125, \quad 5^4 = 625.$$

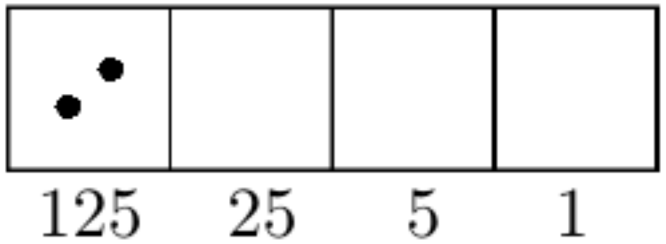
So we know that the left-most box we'll use is the 5^3 box.



How many dots will be in that left-most box? That's the same as asking how many 125s are in 321. Since

$$2 \cdot 125 = 250 \quad \text{and} \quad 3 \cdot 125 = 375,$$

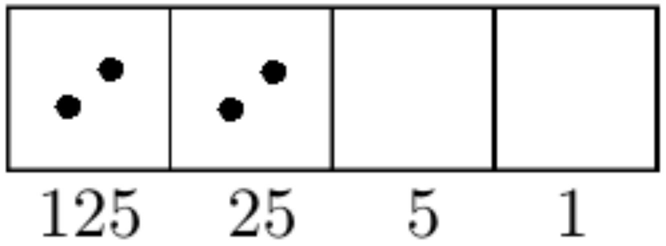
we have two dots in the 5^3 box, representing a total of 250 dots.



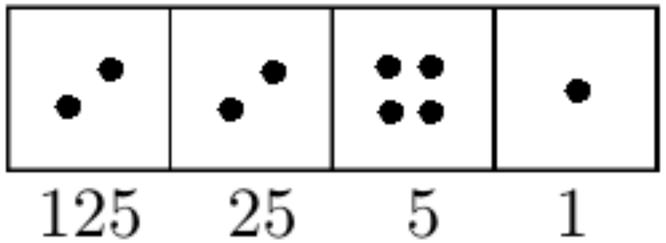
How many dots are left unaccounted for? $321 - 250 = 71$ dots are left.

Now just repeat the process: If we put two in the 5^2 box, that takes care of 50 dots. So far we have two in the 5^3 box and two in the 5^2 box, so that's a total of

$$2 \cdot 125 + 2 \cdot 25 = 300 \text{ dots}.$$



We have 21 dots left to account for. The biggest power of 5 that's less than 21 is just 5. So we can put a “4" in the 5 box, and we have one left over in the one box.



$$2 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 1 = 250 + 50 + 20 + 1 = 321 \text{ dots}.$$

$$\text{So } 321 = 2241_{\text{five}}.$$

The general algorithm to convert from base ten to base b:

1. Start with your base ten number n. Find the largest *power of* b that's less than your number n, say that power is b^k.

2. Figure out how many dots can go in the b^k box without going over the number n. Say that number is a. Put the digit a in the b^k box, and then subtract $n - a \cdot b^k$ to figure out how many dots are left.

3. If your number is now zero, you've accounted for all the dots. Put zeros in any boxes that remain, and you have the number. Otherwise, start over at step (1) with the number of dots you have left.

The method seems a little tricky to describe in complete generality. It's probably better to try a few examples on your own to get the hang of it.

Think/Pair/Share. Use the method above to convert 99_{ten} to base three, to base four, and to base five.

The first method we described fills in the boxes from left to right. Here's another method to convert base ten numbers to another base, and this method fills in the digits from right to left. Again, we'll start with an example and then describe the general method.

Example 4.5 To convert 712 *to a base seven number:*

Divide 712 by seven and find the quotient and remainder:

$$712 \div 7 = 101 \text{ R}5.$$

Put the remainder in the ones place:

$$712 = \underline{???}5_{\text{seven}}.$$

Now take the quotient and divide by seven to find the quotient and remainder:

$$101 \div 7 = 14 \text{ R}3.$$

Put the remainder in the sevens place:

$$712 = \underline{???}35_{\text{seven}}.$$

Take the previous quotient and divide by seven again:

$$14 \div 7 = 2 \text{ R}0.$$

Put the remainder in the 7^2 place:

$$712 = \underline{???}035_{\text{seven}}.$$

Since the quotient that's left is less than seven, it goes in the 7^3 place, and we're done.

$$712 = 2035_{\text{seven}}.$$

Of course, we can (and should!) check our calculation by converting the answer back to base ten:

$$2035_{\text{seven}} = 2 \cdot 7^3 + 0 \cdot 7^2 + 3 \cdot 7 + 5 = 686 + 0 + 21 + 5 = 712_{\text{ten}}\,.$$

So here's a second general method for converting base ten numbers to an arbitrary base b:

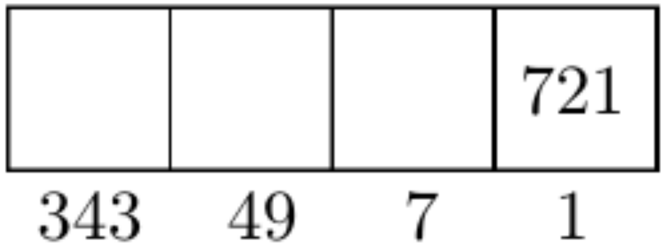
1. Divide the base ten number by b to get a quotient and a remainder.

2. Put the remainder in the right-most space in the base b number.

3. If the quotient is less than b, it goes in the space one spot to the left. Otherwise, go back to step (1) and repeat it with the quotient, filling in the remainders from right to left in the base b number.

We can use the dots and boxes system to explain why this method of quotients and remainders works. It's not just a "trick!" We'll stick with the example of converting 712 to base seven, so we have something specific to talk about.

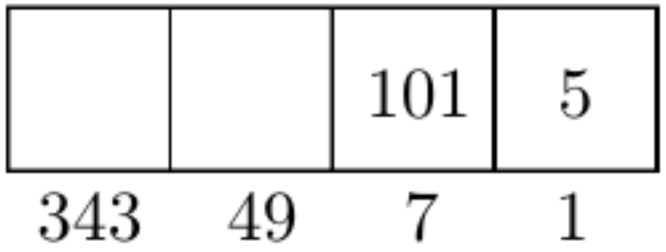
- We imagine 712 dots in the right-most box, since that represents 712 dots total. Since we're converting to base seven, we're in the $1 \leftarrow 7$ system.



- Groups of seven dots will explode, and each group of seven becomes one dot in the next box. How many groups of seven dots are there? Well, there are 101 groups of seven, with 5 dots left over out of a group. That's what we figured out with the calculation

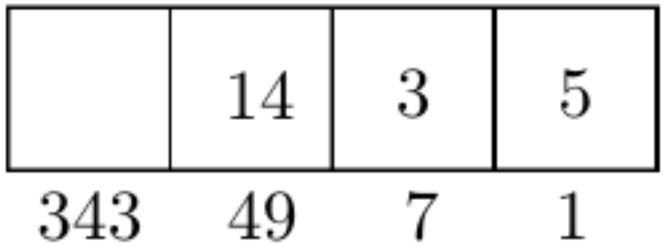
$$712 \div 7 = 101 \text{ R}5.$$

- Imagine we explode all the groups of seven that we can make in the right-most box before we move on. Then we would have 5 dots left in that first box, and 101 dots in the second box.



- Again, groups of seven dots will explode, and each group becomes one dot in the third box. How many groups of seven dots are there? There are 14 groups with three left over. That's what we computed like this:

$$101 \div 7 = 14 \text{ R}3.$$



- OK, now there are 5 dots in the right-most box, 3 dots in the second box, and 14 dots in the third box. We do it all again! Groups of seven explode, and each group forms dot in the next box to the left. Fourteen dots gives two equal groups of seven, none left over.

- So we end up with: 5 dots in the right-most box, 3 dots in the second box, zero dots in the third box, and 2 dots in the fourth box. And there's nothing left to explode!

2	0	3	5
343	49	7	1

- Now we can read off the number left-to-right:

$$712 = 2035_{\text{seven}}\,.$$

Again, the method probably makes more sense if you try it out a few times.

Think/Pair/Share. Use the method described above to convert 250_{ten} to base three, four, five, and six. For each of the computations, write a careful dots-and-boxes explanation for why it works.

Section 5

Number Systems

Definition 5.1. A *positional number system* is one way of writing numbers. It has unique symbols for 1 through b, where b is the *base* of the system. Modern positional number systems also include a symbol for 0. The *positional value* of each symbol depends on its position in the number. The *positional value* of a symbol in the first position is just it's face value. The *positional value* of a symbol in the n^{th} position is b^{n-1} times it's face value. The value of a written number is the sum of the positional values of its digits.

We are already familiar with the numbers in this system as base b numbers. Our number system is a western adaptation of the Hindu–Arabic numeral system developed somewhere between the first and fourth centuries AD. However, numbers have been recorded with tally marks throughout history.

Definition 5.2. In an *additive number system* the value of a written number is the sum of the face values of the symbols which it is composed of. The only symbol necessary for an additive number system is a symbol for 1, however many contain other symbols (typically powers of 10).

Subsection 5.1

History

The ancient Romans and Egyptians used purely additive number systems. The Romans represented the numbers $1, 5, 10, 50, 100, 500,$ and 1000 with I, V, X, L, C, D, and M. So, 2013 would be represented as MMXIII. However, to represent the number one million it would take one thousand M's. For the Egyptians' $0, 1, 10, 100, 1000, 10000, 100000,$ and 1000000 each had their own symbol. For any additive number system very large numbers become impractical to write. However, the Roman numerals did have one efficiency advantage. The order of the symbols mattered. If a symbol to the left was smaller than the symbol to the right, it would be subtracted from it instead of added, i.e nine is represented as IX rather than VIIII.

Think/Pair/Share.

- Is there ambiguity in Roman Numerals?
- What is the maximum number of symbols needed to write any number between 0 and 1,000,000 in Egyptian Numerals?
- What is the maximum number of symbols needed to write any number between 0 and 1,000 in Roman Numerals?

The earliest positional number systems are accredited to the Babylonians (base 60) and the Mayans (base 20). The 60 distinct symbols in the Babylonian's number system were actually composed of the symbols for one and ten used additively. Similarly, the Mayan's used the symbols for one and five additively to compose their symbols for one through twenty. These positional systems were both developed before they had a symbol or a clear concept for zero. In it's stead, a blank space was used. This led to ambiguity. If our system had no zero it would be impossible to tell if the following meant 203, or 2030, or 203000000000000, or two numbers (two and three):

Leonardo Pisano Bigollo,

Leonardo Pisano Bigollo, more commonly known as Fibonacci, played a pivotal role in guiding Europe out of a long period in which the importance and development of math was in marked decline. He was born in Italy circa 1170 CE to Guglielmo Bonacci, a successful merchant. Guglielmo brought his son with him to his merchant post in what is now Algeria and Leonardo was educated in Arabian mathematics. At the time, Roman Numerals dominated Europe and the official means of calculations was the abacus. Although Muḥammad ibn Mūsā al-Khwārizmī had described the use of Hindu-Arabic system through his book *On the Calculation with Hindu Numerals* circa 825 CE, it was neither well received nor widely distributed in Europe until Fibonacci's time.

Fibonacci became a brilliant mathematician and contributed greatly to many fields of mathematics. Of his works, the book *Liber Abaci* arguably had the greatest impact on humanity. In this book he described the Hindu-Arabic system, it's advantages, as well as many business applications. His book was well received and marked the beginning of a reawakening of European mathematics. Although it still took another 300 years for the Hindu-Arabic system to dominate throughout Europe, *Liber Abaci* became the manifesto for the Hindu-Arabic system movement in Europe. The Hindu-Arabic system is now used nearly exclusively throughout the globe.

SECTION 6

EVEN NUMBERS

How do we know if a number is even? What does it mean? Well, some number of dots is *even* if I can divide the dots into pairs, and every dot has a partner.

And some number of dots is *odd* if, when I try to pair up the dots, I always have a single dot left over with no partner.

The number of dots is either even or odd. It's a property of the *quantity* and is doesn't change when you write the number in different bases.

Problem 13. Which of these numbers represent an even number of dots? Explain how you decide.

22_{ten} 319_{ten} 133_{five} 222_{five} 11_{seven} 11_{four}

Think/Pair/Share. Compare your answers to problem 13 with a partner. Then try these together:

a) Count by twos to 20_{ten}.

b) Count by twos to 30_{four}.

c) Count by twos to 51_{seven}.

Think/Pair/Share. You know that you can tell if a number in base 10 is even just by looking at the units digit. Which one of the following statements *best* captures the reason for this rule?

1) It works because even and odd numbers alternate, so you only have to look at the ones place.
2) It works if the number ends with an even digit, but it only works for whole numbers and decimals (e.g. 12 and 1.2).
3) It actually only works if the last digit is 2, 4, 6, or 8.
4) It works because all digits other than the units digit --- for example tens, hundreds, and thousands --- represent even numbers, and sums of even numbers are even.

Problem 14.

a) Write the numbers zero through fifteen in base seven:

$$0_{\text{seven}}, 1_{\text{seven}}, 2_{\text{seven}}, \ldots$$

b) Circle all of the even numbers in your list. How do you know they are even?

c) Find a rule: how can you tell if a number is even when it's written in base seven?

Problem 15.

a) Write the numbers zero through fifteen in base four:

$$0_{\text{four}}, 1_{\text{four}}, 2_{\text{four}}, \ldots$$

b) Circle all of the even numbers in your list. How do you know they are even?

c) Find a rule: how can you tell if a number is even when it's written in base four?

Think/Pair/Share. Discuss your answers to problems 14 and 15.

- Why are the rules for even numbers different in different bases?
- For either your base four rule or your base seven rule, can you explain *why* it works that way?

SECTION 7

ORDERS OF MAGNITUDE

Problem 16. How old were you when you were one million seconds old? (That's 1,000,000.)

- Before you figure it out, write down a guess. What's your gut instinct? About a day? A week? A month? A year? Have you already reached that age? Or maybe you won't live that long?
- Now figure it out! When was / will be your million-second birthday?

Problem 17. How old were you when you were one *billion* seconds old? (That's 1,000,000,000.)

- Again, before you figure it out, write down a guess.
- Now figure it out! When was / will be your billion-second birthday?

Were you surprised by the answers? People (most people, anyway) tend to have a very good sense for small, everyday numbers, but have very bad instincts about big numbers. One problem is that we tend to think *additively*, as if one billion is about a million plus a million more (give or take). But we need to think *mulitplicatively* in situations like this. One billion is $1{,}000 \times$ a million.

So you could have just taken your answer to problem 16 and multiplied it by 1,000 to get your answer to problem 16. Of course, you would probably still need to do some calculations to make sense of the answer.

Think/Pair/Share. When is your one trillion second birthday? What will you do to celebrate?

Think/Pair/Share. The US debt is total amount the government has borrowed. (This borrowing covers the *deficit* --- the difference between what the government spends and what it collects in taxes.) In summer of 2013, the US debt was *on the order of* 10 trillion dollars. (That means more than 10 trillion but less than 100 trillion. If you were to write out the dots-and-boxes picture, the dots would be as far left as the 10,000,000,000 place.)

- If the US pays back one penny every second, will the national debt be paid off in your lifetime? Explain your answer.
- A headline from April 2013 said, “US to Pay Down $35 billion in Quarter 2." Suppose the US pays down $35 billion dollars *every* quarter (so four times per year). About how many years would it take to pay of the total national debt?

Here are some big-number problems to think about. Can you solve them?

Problem 18.

1) Suppose you have a million jelly beans, and you tile the floor with them. How big of an area will they cover? The classroom? A football field? Something bigger? What if it was a billion jelly beans?
2) Suppose you have a million jelly beans and you stack them up. How tall would it be? As tall as you? As a tree? As a skyscraper? What if it was a billion jelly beans? About how many jelly beans (what *order of magnitude*) would you need to stack up to reach the moon? Explain your answers.

SUBSECTION 7.1

FERMI PROBLEMS

James Boswell wrote,"Knowledge is of two kinds. We know a subject ourselves, or we know where we can find information upon it."

But math proves this wrong. There is actually a third kind of knowledge: Knowledge that you *figure out for yourself*. In fact, this is what scientists and mathematicians do for a living: they create new knowledge! Starting with what is already known, they ask "what if … " questions. And eventually, they figure out something new, something no one ever knew before!

Even for knowledge that you *could* look up (or ask someone), you can often figure out the answer (or a close approximation to the answer) on your own. You need to use a little knowledge, and a little ingenuity.

Fermi problems, named for the physicist Enrico Fermi, involve using your knowledge, making educated guesses, and doing reasonable calculations to come up with an answer that might at first seem unanswerable.

Example 7.1. Here's a classic Fermi problem: How many elementary school teachers are there in the state of Hawaii?

You might think: How could I possibly answer that? Why not just google it? (But some Fermi problems we meet will have — gasp! — non-googleable answers.)

First let's define our terms. We'll say that we care about classroom teachers (not administrators, supervisors, or other school personnel) who have a permanent position (not a sub, an aide, a resource room teacher, or a student teacher) in a grade K–5 classroom.

But let's stop and think. Do you know the population of Hawaii? It's about 1,000,000 people. (That's not exact, of course. But this is an exercise is estimation. We're trying to get at the *order of magnitude* of the answer.)

How many of those people are elementary school students? Well, what do you know about the population of Hawaii? Or what do you *suspect* is true? A reasonable guess would be that the population is evenly distributed across all age groups. Something like this? We'll assume people don't live past 80. (Of course some people do! But we're all about making simplifying assumptions right now. That gives us 8 age categories, with about 125,000 people in each category.

age range	# people
0 – 9	125,000
10 – 19	125,000
20 – 29	125,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	125,000
70 – 79	125,000

An even better guess (since we have a large university that draws lots of students) is that there's a “bump" around college age. And some people live past 80 , but there are probably fewer people in the older age brackets. Maybe the breakdown is something like this? (If you have better guesses, use them!)

age range	# people
0 – 9	125,000
10 – 19	130,000
20 – 29	140,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	120,000
> 70	105,000

So, how many K–5 students are in Hawaii? That covers six years of the 0–9 (maybe 10) range. If we are still going with about the same number of people at each age, there should be about 12,500 in each grade for a total of $12{,}500 \times 6 = 75{,}000$ K–5 students.

OK, but we really wanted to know about K–5 *teachers*. One nice thing about elementary school: there tends to be just one teacher per class. So we need an estimate of how many classes, and that will tell us how many teachers.

So, how many students in each class? It probably varies a bit, with smaller kindergarten classes (since they are more rambunctious and need more attention), and larger fifth grade classes. There are also smaller classes in private schools and charter schools, but larger classes in public schools. A reasonable average might be 25 students per class across all grades K–5 and all schools.

So that makes $75{,}000 \div 25 = 3{,}000$ K–5 classrooms in Hawaii. And that should be the same as the number of K–5 teachers.

Problem 19. How good is this estimate? Can you think of a way to check and find out for sure?

So now you see the process for tackling a Fermi problem:

- Define your terms.
- Write down what you know.
- Make some reasonable guesses / estimates.
- Do some simple calculations.

Try your hand at one or more of these:

Problem 20. How much money does UH Manoa earn in parking revenue each year?

Problem 21. How many tourists visit Waikiki in a year?

Problem 22. How much gas would be saved in Hawaii if one out of every ten people switched to a carpool?

Problem 23. How high can a climber go up a mountain on the energy in one chocolate bar?

Problem 24. How much pizza is consumed by UH Manoa students in a month?

Problem 25. How much would it cost to provide free day care to every 4th grader in the US?

Problem 26. How many books are in Hamilton library?

Problem 24. Make up your own Fermi problem… what would you be interested in calculating? Then try to solve it!

Section 8

Problem Bank

Problem 28.

a) If you were counting in base four, what number would you say just before you said 100_{four}?

b) What number is one more than 133_{four}?

c) What is the greatest three-digit number that can be written in base four? What numbers come just before and just after that number?

Problem 29. Explain what is wrong with writing 313_{two} or 28_{eight}.

Problem 30.

a) Write out the base three numbers from 1_{three} to 200_{three}.

b) Write out the base five numbers from 1_{five} to 100_{five}.

c) Write the four base six numbers that come after 154_{six} .

Problem 31. Convert each base-10 number to a base-4 number. Explain how you did it.

13 8 24 49

Challenges: 0.125 $0.111... = 0.\overline{1}$

Problem 32. In order to use base sixteen, we need sixteen digits — they will represent the numbers zero through fifteen. We can use our usual digits 0–9, but we need *new symbols* to represent the *digits* ten, eleven, twelve, thirteen, fourteen, and fifteen. Here's one standard convention:

base 10 number	base 16 digit
10	A
11	B
12	C
13	D
14	E
15	F

a) Convert these numbers from base sixteen to base ten, and show your work:

$6D_{\text{sixteen}}$ AE_{sixteen} $9C_{\text{sixteen}}$ $2B_{\text{sixteen}}$

b) Convert these numbers from base ten to base sixteen, and show your work:

97 144 203 890

Problem 33. How many different symbols would you need for a base twenty-five system? Justify your answer.

Problem 34. All of the following numbers are multiples of three.

$$3, \quad 6, \quad 9, \quad 12, \quad 21, \quad 27, \quad 33, \quad 60, \quad 81, \quad 99.$$

a) Identify the *powers of* 3 in the list. Justify your answer.
b) Write each of the numbers above in base three.
c) In base three: how can you recognize a *multiple of* 3? Explain your answer.
d) In base three: how can you recognize a *power of* 3 ? Explain your answer.

Problem 35. All of the following numbers are multiples of five.

$$5, \quad 10, \quad 15, \quad 25, \quad 55, \quad 75, \quad 100, \quad 125, \quad 625, \quad 1000.$$

a) Identify the *powers of* 5 in the list. Justify your answer.
b) Write each of the numbers above in base five.
c) In base five: how can you recognize a *multiple of* 5? Explain your answer.
d) In base five: how can you recognize a *power of* 5? Explain your answer.

Problem 36. Convert each number to the given base.

a) $395_{\mathbf{ten}}$ into base eight.
b) 52_{ten} into base two.
c) 743_{ten} into base five.

Problem 37. What bases makes theses equations true? Justify your answers.

a) $35 = 120\_$
b) $41_{\text{six}} = 27\_$
c) $52_{\text{seven}} = 34\_$

Problem 38. What bases makes theses equations true?

a) $32 = 44\_$
b) $57_{\text{eight}} = 10\_$
c) $31_{\text{four}} = 11\_$
d) $15_x = 30_y$

Problem 39.

a) Find a base ten number that is twice the product of its two digits. Is there more than one answer? Justify what you say.
b) Can you solve this problem in any base other than ten?

Problem 40.

a) I have a four-digit number written in base ten. When I multiply my number by four, the digits get reversed. Find the number.
b) Can you solve this problem in any base other than ten?

Problem 41. Convert each base-4 number to a base-10 number. Explain how you did it.

$$13_{\text{four}} \qquad 322_{\text{four}} \qquad 101_{\text{four}} \qquad 1300_{\text{four}}$$

Challenges: 0.2_{four} $\qquad 0.111\ldots = 0.\overline{1}_{\text{four}}$

Problem 42. Consider this base ten number (I got this by writing the numbers from 1 to 60 in order next to one another):

$$12345678910111213\ldots57585960$$

a) What is the largest number that can be produced by erasing one hundred digits of the number? (When you erase a digit it goes away. For example, if you start with the number 12345 and erase the middle digit, you produce the number 12345.) How do you *know* you got the largest possible number?

b) What is the smallest number that can be produced by erasing one hundred digits of the number? How do you *know* you got the smallest possible number?

Problem 43. Can you find numbers (not necessarily single digits!) a and b so that $a_b = b_a$? Can you find more than one solution? What must be true of a and b? Justify your answers.

SECTION 9

EXPLORATION

Problem 44. Jay decides to play with a system that follows a $1 \leftarrow 1$ rule. He puts one dot into the right-most box. What happens?

Problem 45. Poindexter decides to play with a system that follows the rule $2 \leftarrow 3$.

a) Describe what this rule does when there are three dots in the right-most box.

b) Draw diagrams or use buttons or pennies to find the $2 \leftarrow 3$ codes for the following numbers:

1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,24,27,30,33,36, and 39

Can you find (and *explain*) any patterns?

Problem 46. Repeat problem 45 for your own rule. Choose two numbers a and b and figure out what the code is for your $a \leftarrow b$ system for each of the numbers above.